JOURNAL OF COMPUTATIONAL PHYSICS **121**, 185–189 (1995)

NOTE

On the Calculation of the Gravitational Force of Axi-symmetric Infinitely Thin Disks

1. INTRODUCTION

We describe an efficient method for the numerical calculation of the gravitational force of axi-symmetric infinitely thin disks. The evolution of a self-gravitating astrophysical disk is affected by the force acting in the plane of the disk. Because of the long evolution times, the calculation of the dynamics of such a disk requires high numerical accuracy of the intermediate computations. Numerical codes which correctly satisfy the conservation of mass and angular momentum are used (see, e.g., Norman *et al.* [1]). The gravitational force enters the dynamical equations via the source terms and must be calculated with sufficient precision. We present a computational scheme which meets the requirements of numerical accuracy, computational efficiency, and vectorization. The most crucial point is the regularization of the singularity of the integral kernel. The separate treatment of the singular point commonly used is avoided. Our numerical scheme is based on piecewise approximations of the surface density by parabolas. The time consuming computation of the elliptical integrals is omitted by a matrix representation of the integral operator. The problem of the singularities of the integrands is removed by a particular representation of the approximation parabolas. Since astrophysical disks are often not isolated, but surrounded by exterior masses, some exterior surface density distributions are included.

2. THE INTEGRAL REPRESENTATION OF THE GRAVITATIONAL FORCE

For an axi-symmetric disk, the angular integrations can be performed so that the potential and the force can be represented by single integrals over the radial coordinate. The integrands can be written in terms of the elliptical integrals **K** and **E**. Let $\sigma(s)$ be the surface density and $\phi(s)$ the gravitational potential in the plane of the disk, where s is the distance from the axis of symmetry. We denote the polar coordinates of the source point by (r, φ). Then

$$\phi(s) = -G\int_0^\infty \sigma(r)r\,dr\int_0^{2\pi}\frac{d\varphi}{(s^2+r^2-2\,sr\cos\varphi)^{1/2}}. \quad (1)$$

Performing the angular integration one obtains

$$\phi(s) = -4G\int_0^\infty \sigma(r)K(r,s)\,dr, \quad (2)$$

where the kernel K is given by

$$K(r,s) = \frac{r}{s}\mathbf{K}\left(\frac{r}{s}\right) \quad \text{for } r<s,$$
$$K(r,s) = \mathbf{K}\left(\frac{s}{r}\right) \quad \text{for } r>s. \quad (3)$$

Differentiating the potential one obtains

$$\frac{\partial}{\partial s}\phi(s) = 4G\int_0^\infty \sigma(r)H(s,r)\,dr, \quad (4)$$

where the kernel H is

$$H(s,r) = \frac{r\mathbf{E}(r/s)}{(s^2-r^2)} \quad \text{for } r<s, \quad (5)$$

$$H(s,r) = \frac{1}{s}\left\{\frac{r^2\mathbf{E}(s/r)}{(s^2-r^2)} + \mathbf{K}(s/r)\right\} \quad \text{for } s<r. \quad (6)$$

The Poisson integral (2) has only a logarithmic singularity, the force integral (4) a pole in addition. However, for numerical and physical reasons, we calculate the force integral. In the case of numerical simulations of dynamical problems with a large number of grid points and repeated calculations of the gravitational force, the time consuming computation of the elliptical integrals must be avoided. This requirement is fulfilled by a scheme based on a matrix representation of the integral operator (see below).

The surface density $\sigma(r)$ must be continuous on $0 \le r < \infty$. We assume that σ is given on $0 \le r \le R$ with either $\sigma(r) \to 0$ for $r \to R$ so that $\sigma(r) = 0$ for $r \ge R$, or with $\sigma(R) = \mu(R)$, where $\mu(r)$ is a given exterior surface density:

$$\mu(r) = \sigma(R)\frac{r^n}{R^n} \quad \text{for } r \ge R;\ n = 0, -1, -3, -5. \quad (7)$$

0021-9991/95 $12.00

Evaluating the integrals (20) and (21) for $i = 1, ..., k-1$, we get

$$W_{i,k} = \frac{s^2}{3}\{(1 - x_{i+1}^2)\mathbf{K}(x_{i+1}) - (1 + x_{i+1}^2)\mathbf{E}(x_{i+1}) - (1 - x_i^2)\mathbf{K}(x_i) + (1 + x_i^2)\mathbf{E}(x_i)\}, \tag{31}$$

and for $i = k, ..., N-1$ we get

$$W_{i,k} = \frac{s^2}{3}\{-[(1 - y_{i+1}^2)\mathbf{K}(y_{i+1}) - (1 + y_{i+1}^2)\mathbf{E}(x_{i+1})]y_{i+1}^{-3} + [(1 - y_i^2)\mathbf{K}(y_i) - (1 + y_i^2)\mathbf{E}(y_i)]y_i^{-3}\}. \tag{32}$$

Now let us briefly discuss the calculation of the elements $V_{i,k}$. Integrations by parts can be performed to obtain integrals which are of a more elementary form. However, then severe numerical cancellation can occur. Therefore we calculate the original integrals (18) and (19). An adaptive Romberg integration tackles the problem of the (weak) divergence of the derivatives of the term $(1 - y)\mathbf{K}(y)$.

Now we present the contributions of the exterior surface density. For all field points s_k with $k = 2, ..., N-1$ this contribution is given by Eq. (24). We put $z = x_k = s_k/R$. The expansions for $z \to 0$ should be used if $z < 10^{-4}$.

For $n = 0, -1, -3, -5$ we get

$$U_{N,k} = \frac{\mathbf{E}(z) - \mathbf{K}(z)}{z} \to -\frac{\pi}{4}z\left(1 + \frac{3}{8}z^2\right) \quad \text{for } z \to 0, \tag{33}$$

$$U_{N,k} = \frac{\pi/2 - \mathbf{K}(z)}{z} \to -\frac{\pi}{8}z\left(1 + \frac{9}{16}z^2\right) \quad \text{for } z \to 0, \tag{34}$$

$$U_{N,k} = \{-(2 - z^2)\mathbf{K}(z) + 2\mathbf{E}(z)\}z^{-3} \to -\frac{\pi}{16}z\left(1 + \frac{3}{4}z^2\right) \quad \text{for } z \to 0, \tag{35}$$

$$U_{N,k} = \left\{-\left[z^4 + \frac{4}{9}(3z^2 + 4)(1 - z^2)\right]\mathbf{K}(z) + \frac{4}{9}(z^2 + 4)\mathbf{E}(z)\right\}z^{-5} \to -\frac{\pi}{24}z\left(1 + \frac{27}{32}z^2\right) \quad \text{for } z \to 0. \tag{36}$$

At $s_N = R$ the contribution of the exterior surface density is given by the element $U_{N,N}$ which is obtained from Eqs. (25) and (30).

For $n = 0, -1, -3, -5$ we have

$$U_{N,N} = 0, \pi/2 - 1, 1, 7/9. \tag{37}$$

6. DISCUSSION

The elements of the matrices U, V, W can be precalculated with sufficient numerical accuracy. We find that numerical cancellations are insignificant. Also the cancellation of positive and negative force contributions plays no role. The code has been tested by use of surface density distributions with known analytical solutions of the Poisson integral. We have calculated the force of disks with finite radii and of infinitely extended disks with an asymptotic behavior given by the expression (8). (Ten relevant disks with analytically known result have been used.) Both equidistant grids and grids with a geometrical progression of the intervals have been adapted. Surface densities with peak-like behavior at the origin have been used as well as oscillating surface densities. The tests have shown that an accuracy up to 10^{-8} can be achieved. The accuracy of the gravitational force is determined only by the accuracy of the approximation of the surface density by the parabolas.

There are also astrophysical questions which imply the gravitational potential instead of, or in addition to, the force. For instance, the equilibrium equation of a rotating barotropic disk can be integrated and written in terms of potentials. In such cases the potential can be calculated readily from the force by use of weighted parabolas (Schmitz [8]).

The presented method for the calculation of the force (4) can be applied also to the potential (2). Since the scheme (13) and the transformation (17) are unchanged, computational time is saved only at the preceding computation of the matrix elements. For this reason and because of the problems caused by numerical differentiation we prefer the direct calculation of the force integral.

A simple example for the method described in this note is the computation of the force of an infinitely thin sheet the structure of which depends only on ξ, one of the two Cartesian coordinates which define the plane of the sheet

$$\frac{\partial}{\partial \xi}\phi(\xi) = 2G\int_{-\infty}^{+\infty}\frac{\sigma(\xi')}{\xi - \xi'}d\xi'. \tag{38}$$

Apart from a factor $(2\pi G)^{-1}$ this is the Hilbert transform of the surface density σ. We assume that

$$\sigma(\xi') = 0 \quad \text{for } -\infty < \xi' \le 0; L \le \xi' < \infty. \tag{39}$$

Let $\{\xi_i', i = 1, N; \xi_1' = 0; \xi_N' = L\}$ be a set of space points, and let σ_i the numerical values of the surface density. Then the matrix elements are given by

$$U_{i,k} = \frac{1}{2}\ln\left|\frac{\xi_k - \xi_i}{\xi_k - \xi_{i+1}}\right| \quad \text{for } i \ne k-1; i \ne k, \tag{40}$$

$$U_{k-1,k} = \tfrac{1}{2}\ln|\xi_k - \xi_{k-1}|, \quad U_{k,k} = -\tfrac{1}{2}\ln|\xi_k - \xi_{k+1}|, \tag{41}$$

$$V_{i,k} = \tfrac{1}{2}(\xi_i - \xi_{i+1}), \tag{42}$$

$$W_{i,k} = \tfrac{1}{4}(2\xi_k - \xi_{i+1} - \xi_i)(\xi_{i+1} - \xi_i). \tag{43}$$

The representation of the surface density $\sigma(\xi) = \xi(1 - \xi)$ for $0 \le \xi \le 1$ by parabolas is exact. Comparing the exact result $\partial\phi(\xi)/\partial\xi = 2G\{\xi - \frac{1}{2} + \xi(\xi - 1)\ln((\xi - 1)/\xi)\}$ with the numerical result we find that for $N = 100$ to 500 grid points and a machine accuracy 10^{-16}, a relative accuracy $\varepsilon = 10^{-12}$ to 10^{-14} can be achieved. The largest value $\varepsilon \approx 10^{-12}$ at $\xi \approx 0.5$ is due to the cancellation of opposite contributions. This example shows that cancellation errors are completely insignificant as compared to approximation errors.

Clearly, the code is written for a vector machine. It consists of vector instructions with vectors of length N. With respect to the CPU-time, the main parts of the code are the transformation (11) and the algebraic operations (13). The operations are equivalent to $10N$ elementary vector instructions for vectors of length N. The calculation of the coefficients of the parabolas needs only 30 elementary vector operations. Calculations have been performed on a two pipeline CDC Cyber 205 (University of Karlsruhe), A cray Y-MP EL (University of Würzburg), and a Cary Y-MP 8/864 (Forschungszentrum Jülich). We find that the measured CPU-times correspond to the predicted times (tact-frequency multiplied by the number of operations). For $N = 500$, the CPU-time for the calculation of the force at all points amounts to less than 10 ms for both the Cyber 205 and the Y-MP 8/864 and 30 ms for the Y-MP EL.

REFERENCES

1. M. L. Norman, J. R. Wilson, and R. T. Barton, *Astrophys. J.* **239**, 968 (1992).
2. L. Mestel, *Mon. Not. R. Astron. Soc.* **126**, 553 (1963).
3. R. L. Kurucz, SAO Spec. Rep. 309, 1973 (unpublished).
4. A. Harten, B. Enquist, S. Osher, and S. Chakravarthy, 1987, *J. Comput. Phys.* **71**, 231 (1987).
5. R. Le Veque, *Numerical Methods for Conservation Laws* (Birkhäuser, Basel, 1992).
6. R. Hammer and P. Ulmschneider, *Astron. Astrophys.* **65**, 273 (1978).
7. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, London, 1980).
8. F. Schmitz, *Astron. Astrophys.* **230**, 509 (1990).

Received August 17, 1993; revised November 28, 1994

F. Schmitz
M. Kaisig

Astronomisches Institut
Universität Würzburg
Am Hubland
D-97074 Würzburg
Germany